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Integral Equation Procedures for Eddy Current Problems*

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Two methods are presented for the solution of a class of two dimensional eddy current problems. The specific problem considered is that of a wire carrying a periodic current parallel to a uniform conducting cylinder. Both methods yield integral equations over the boundary of the conductor. The first method is exact while the second is an asymptotic procedure valid for a certain parameter range and reflecting the skin effect. Numerical implementation procedures are given for both methods and some numerical results are presented.

1. INTRODUCTION

The problem under consideration is the determination of the effect of metallic obstacles on electromagnetic fields in air. The obstacles distort the incident field and also support eddy currents which produce power losses.

We consider a very special but important class of these problems and present two methods of solution. The first method is theoretically exact and would apply equally well to dielectric obstacles. The second is approximate and applies for a certain parameter range. It is based on the skin effect and thus applies only for metal. We give numerical procedures for both methods and present the results of numerical experiments.

We neglect conduction current in air and displacement current in the metal and we assume $\mathbf{B} = \mu_m \mathbf{H}$ in the metal. In air we have $\mathbf{B} = \mu_a \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$ and we note that for non-ferromagnetic materials $\mu_a \approx \mu_m$. The conduction current in the metal will be $\mathbf{J} = \sigma \mathbf{E}$.

We make two simplifications. First we treat fields which are time periodic with a single angular frequency ω. The major assumption is that the fields are transverse

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magnetic. This means that, with a proper choice of x, y, z axes, $\mathbf{E} = E\hat{k}$, $\mathbf{H} = H^1\hat{\imath} + H^2\hat{\jmath}$, where E, H^1 and H^2 are functions of x and y only. The prototype example, and the one we treat, is that of a uniform cylindrical conductor in the field of a wire parallel to the cylinder and carrying a periodic current.

Scaling is very important for us. If L is a representative length then there are two basic non-dimensional parameters:

$$\alpha = \sqrt{\omega\mu_m\sigma}\, L, \qquad \beta = \sqrt{\mu_a\varepsilon}\, \omega L. \tag{1.1}$$

A common engineering approximation is to neglect displacement current in air which means taking $\beta = 0$. In *MKS* units $\mu_a\varepsilon$ is about 3×10^{-9} so this would seem to be a reasonable approximation for moderate frequencies. A second approximation is to assume infinite conductivity of the metal, $\alpha = \infty$. For copper $\sqrt{\mu_m\sigma}$ is about 3 so this approximation may or may not be valid. For iron, however, $\sqrt{\mu_m\sigma}$ is of the order of 100 so α will typically be large.

In both our methods there is a modification and simplification when $\beta = 0$. Our second method is an asymptotic procedure for large α, the first term of which is the infinite conductivity approximation. Numerical experiments indicate that this asymptotic procedure yields quite good results for values of α which are not very big thus bringing us well within the practical range. When it is valid the second method represents a major simplification.

Several numerical procedures for eddy current problems have been developed in recent years. Many of these methods are described in [3]. Some of the methods work directly with the differential equations and apply finite elements. Theoretical results on procedures of this type are described in [2, 6, 8, 9] and there are many papers on applications to specific problems.

Our methods belong to the category of boundary integral equations methods. Here too there have been many methods proposed. We want to give a brief discussion of some of the results and their connections with our work. First we mention the extensive discussion in [13]. The problem treated there corresponds to the infinite conductivity situation, a boundary value problem. It is done completely, in three dimensions. Two dimensional versions of the perfect conductor problems are discussed rigorously in [5, 7]. A comprehensive discussion of numerical implementation of the method of [13] appears in [10].

For the interface problems one finds integral equation procedures discussed in [12, 15] with remarks on numerical implementations. There is a difference between our method and those of [12, 15]. The latter are based on the use of Green's theorem and, as a consequence, one has to compute the normal derivative of a double layer potential in these. Our method avoids this difficulty.

We have devoted considerable effort in our work to a careful verification that our equations do have solutions. A feature of our work, which is new in so far as we know, is the treatment of the case in which there is no displacement current outside. This requires significant modifications. We believe our methods are, in fact, new and rather easily implemented. In particular we do not know of another statement of the asymptotic procedure we present. We remark, however, that the ideas of that procedure are present in [16], where the special case of a half-space problem was treated.

We wish to thank the referees for calling our attention to Refs. [10, 12, 15, 16].

The methods presented here appear capable of extension to three dimensional problems. They cannot, however, treat inhomogenous or non-linearly magnetic materials.

2. Statement of the Problem

We describe more precisely the problem indicated above. We have an infinite conducting cylinder of uniform cross section parallel to the z-axis. We divide position coordinates by a cross sectional length to obtain dimensionless variables x and y. We let Ω denote the cross section of the cylinder in the $x-y$ plane, Γ its boundary and Ω^+ the region outside (Fig. 1).

We suppose there is a wire[1] parallel to the z-axis through $(x_0, 0)$ carrying a current $I(t)$ of the form,

$$I(t) = \mathrm{Re}(I_0 e^{-i\omega t}). \tag{2.1}$$

Following the outline of the Introduction we assume that the electric and magnetic fields have the form,

$$\begin{aligned} \mathscr{E}(x, y, z, t) &= \mathrm{Re}(E(x, y)\,\hat{k} e^{-i\omega t}), \\ \mathscr{H}(x, y, z, t) &= \mathrm{Re}((H_1(x, y)\hat{\imath} + H_2(x, y)\hat{\jmath})\, e^{-i\omega t}). \end{aligned} \tag{2.2}$$

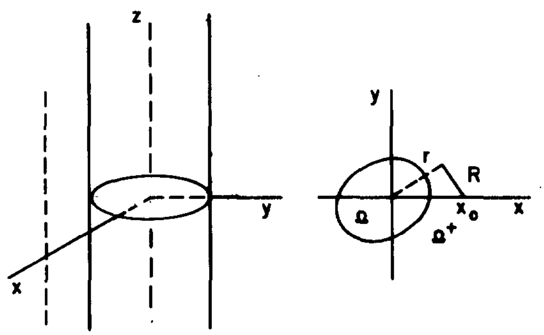

Figure 1

[1] The case of several wires could also be treated.

In particular the field due to the wire will have form (2.2) with

$$
\begin{aligned}
E^0(x, y) &= \omega\mu_a I_0 \varphi_\beta^0(x, y), & H_1^0(x, y) &= \frac{I_0}{L}\varphi_{\beta,y}^0(x, y), \\
H_2^0(x, y) &= -\frac{I_0}{L}\varphi_{\beta,x}^0(x, y), & \varphi_\beta^0(x, y) &= -\frac{i}{4}H_0^{(1)}(\beta R).
\end{aligned} \tag{2.3}
$$

Here R is as in Fig. 1 and $H_0^{(1)}$ is the Hankel function of first kind and order zero.

Fields (2.3) satisfy Maxwell's equations in air. We seek the total fields in the same form, that is,

$$
E = \omega\mu I_0 \varphi, \qquad H_1 = \frac{I_0}{L}\varphi_y, \qquad H_2 = -\frac{I_0}{L}\varphi_x, \tag{2.4}
$$

where φ is again a function of x and y and $\mu = \mu_a(\mu_m)$ in air (metal). Fields (2.4) are automatically divergence free and they will satisfy the remaining Maxwell equation if,

$$
\begin{aligned}
\Delta\varphi &\equiv \varphi_{xx} + \varphi_{yy} = -\beta^2\varphi && \text{in } \Omega^+ \text{ (air)}, \\
\Delta\varphi &= -i\alpha^2\varphi && \text{in } \Omega \text{ (metal)}.
\end{aligned} \tag{2.5}
$$

The transition conditions across the air–metal interface are that the tangential components of electric and magnetic fields be continuous. For (2.4) this implies,

$$
\mu_a\varphi^+ = \mu_m\varphi^-, \qquad \left(\frac{\partial\varphi}{\partial n}\right)^+ = \left(\frac{\partial\varphi}{\partial n}\right)^- \qquad \text{on } \Gamma, \tag{2.6}
$$

where the plus and minus denote limits from Ω^+ and Ω, respectively. $E - E^0$ and $H - H^0$ are the scattered fields so we impose the condition:

$$
\varphi - \varphi_\beta^0 \text{ satisfies a radiation condition.} \tag{2.7}
$$

The problem we consider is specified by (2.5)–(2.7).[2] We call this problem (P_β). As indicated in the Introduction we want to consider also the case $\beta = 0$. A little care is necessary. We recall that the Hankel function $H_0^{(1)}$ has the expansion, [1],

$$
-\frac{i}{4}H_0^{(1)}(z) = \frac{1}{2\pi}\log z + \lambda_0 + \sum_{k=1}^{\infty} a_k z^{2k}\log z + \sum_{k=1}^{\infty} b_k z^{2k}. \tag{2.8}
$$

(An observation that is of importance later on is that both real and imaginary parts of λ_0 are non-zero.) Formula (2.8) shows that

$$
\varphi_\beta^0(x, y) = -\frac{i}{4}H_0^{(1)}(\beta R) = \frac{1}{2\pi}\log R + \left(\frac{1}{2\pi}\log\beta + \lambda_0\right) + O(\beta^2\log\beta) \tag{2.9}
$$

[2] φ_β^0 is singular at $(x^0, 0)$ so $(2.5)_1$ is to hold only for $(x, y) \neq (x_0, 0)$.

as $\beta \to 0$. Thus there is a question as to what to do as $\beta \to 0$. It turns out that the appropriate thing to do is to keep only the first term on (2.9). We set

$$\varphi_0^0(x, y) = \frac{1}{2\pi} \log R. \tag{2.10}$$

Then the correct problem for $\beta = 0$ is to make $\varphi - \varphi_0^0$ regular and to require

$$\Delta\varphi = 0 \qquad \text{in } \Omega^+, \qquad \Delta\varphi = -i\alpha^2\varphi \qquad \text{in } \Omega. \tag{2.11}$$

We also need to drop the radiation condition and replace it by,

$$\varphi \text{ bounded as } r = \sqrt{x^2 + y^2} \to \infty. \tag{2.12}$$

We call (2.11), (2.12) and (2.6) problem (P_0).

Remark 2.1. If the obstacles are dielectric rather than metal the only change in the problem is that the right side of $(2.5)_2$ is replaced by $-\mu_D \varepsilon_D \omega^2 L^2 \varphi$, where ε_D and μ_D are permittivity and permeability for the dielectric.

Remark 2.2. We have stated that (P_0) is the correct limiting problem. This assertion is justified in [4], where it is shown that the (unique) solution of (P_β) for $\beta > 0$ tends to that of (P_0) as $\beta \to 0$. This proof is quite complicated and we do not include it although it follows from the ideas developed in Section 6.

Remark 2.3. The eddy current density $\mathscr{T}$ is equal to $\sigma\mathscr{E}$. Hence we have

$$\mathscr{T}(x, y, z, t) = \operatorname{Re}(\omega\mu_m \sigma I_0 \varphi(x, y)\, \hat{k} e^{-i\omega t}). \tag{2.13}$$

The total current $\mathscr{T}(t)$ flowing in the conductor is obtained by integrating. Recalling that the spatial valuables were divided by L and that $\omega\mu_m \sigma L^2 = \alpha^2$ this yields, from (2.13),

$$\mathscr{T}(t) = \operatorname{Re}\left(\alpha^2 I_0 \int_\Omega \int \varphi(x, y)\, dx\, dy\, e^{-i\omega t}\right). \tag{2.14}$$

3. The Integral Equation Method

In order to describe and analyze our procedures we need some notation. First we need a parametric representation of Γ. It is convenient to use $[0, 2\pi]$ as a parameter range. Thus we assume,

$$\Gamma\colon \mathbf{x} = \mathbf{X}(\tau), \qquad 0 \leqslant \tau \leqslant 2\pi, \qquad 0 \neq p(\tau) \equiv \|\dot{\mathbf{X}}(\tau)\|. \tag{3.1}$$

(Here and in what follows $\mathbf{x} = (x, y)$ and $\|\mathbf{x}\|^2 = x^2 + y^2$.) In order to avoid technical detail we assume $\mathbf{X}$ is analytic.

We let $\mathscr{F}$ denote the space of continuous 2π-periodic functions and write Mf for the mean-value of a function in $\mathscr{F}$,

$$Mf = (2\pi)^{-1} \int_0^{2\pi} f(\tau)\, d\tau. \tag{3.2}$$

We will be concerned with integral operators on $\mathscr{F}$. If $k(\sigma, \tau)$ is a square integrable, doubly 2π-periodic function we define,

$$Kf(\sigma) \equiv (k \wedge f)(\sigma) = \int_0^{2\pi} k(\sigma, \tau)\, f(\tau)\, d\tau. \tag{3.3}$$

K is then a linear map from $\mathscr{F}$ into itself.

We will have to deal with a class of integral operators whose kernels are logarithmically infinite. We write,

$$\begin{gathered} \rho(\sigma, \tau) = \|\mathbf{X}(\sigma) - \mathbf{X}(\tau)\|, \qquad g_0(\sigma, \tau) = (2\pi)^{-1} \log \rho(\sigma, \tau), \\ \ell(\sigma, \tau) = (2\pi)^{-1} \log \sin \left| \frac{\sigma - \tau}{2} \right|. \end{gathered} \tag{3.4}$$

We see that ℓ is jointly 2π-periodic and it is not difficult to verify that for $\mathbf{X}$ analytic one has,

$$g_0(\sigma, \tau) = \ell(\sigma, \tau) + m(\sigma, \tau), \tag{3.5}$$

where m is periodic and analytic. We write, as above,

$$G_0 f = g_0 \wedge f \qquad \text{and} \qquad Lf = \ell \wedge f.$$

Next we introduce the fundamental singularities on $\Delta n = -\gamma^2 n$:

$$\mathscr{G}_\gamma(r) = -\frac{i}{4} H_0^{(1)}(\gamma r)\ \gamma \neq 0, \qquad \mathscr{G}_0(r) = \frac{1}{2\pi} \log r. \tag{3.6}$$

From expansion (2.8) we have,

$$\mathscr{G}_\gamma(z) = \mathscr{G}_0(z) + \lambda_0 + \mathscr{H}_\gamma(z), \tag{3.7}$$

where $\mathscr{H}_\gamma$ has a second derivative of $O(\log z)$ and $\mathscr{H}_\gamma = O(\gamma^2 \log \gamma)$ as $\gamma \to 0$. We use $\mathscr{G}_\gamma$ to construct the simple layer potentials $\mathscr{U}_\gamma$,

$$\mathscr{U}_\gamma f(\mathbf{x}) = \int_0^{2\pi} f(\tau)\, \mathscr{G}_\gamma(\|\mathbf{x} - \mathbf{X}(\tau)\|)\, d\tau, \tag{3.8}$$

for $f \in \mathscr{F}$.

The properties of $\mathscr{U}_\gamma$ are well known. $\mathscr{U}_\gamma$ is a solution of $\Delta v = -\gamma^2 v$ in Ω and in Ω^+. It is continuous in all space and its value on Γ is an integral operator,

$$\mathscr{U}_\gamma f(\mathbf{X}(\sigma)) \equiv G_\gamma f(\sigma) \equiv (g_\gamma \wedge f)(\sigma), \qquad g_\gamma(\sigma, \tau) = \mathscr{G}_\gamma(\rho(\sigma, \tau)). \tag{3.9}$$

From (3.4)–(3.7) we have,

$$g_\gamma(\sigma, \tau) = \ell(\sigma, \tau) + r_\gamma(\sigma, \tau), \qquad G_\gamma = L + R_\gamma, \tag{3.10}$$

where r_γ has second derivatives of order $\log|\sigma - \tau|$.

The normal derivatives of $\mathscr{U}_\gamma$ jump across Γ according to the formula,

$$\left(\frac{\partial}{\partial n}\mathscr{U}_\gamma f\right)^{\pm}(\mathbf{X}(\sigma)) = \pm\frac{1}{2}\frac{f(\sigma)}{p(\sigma)} + N_\gamma f(\sigma). \tag{3.11}$$

Here $N_\gamma f = n_\gamma \wedge f$, where n_γ is continuous.

We can now describe our solution procedure, first for (P_β) with $\beta > 0$. Here we seek a solution in the form,

$$\begin{aligned} \varphi(\mathbf{x}) &= \mathscr{U}_{\sqrt{i\alpha}} f(\mathbf{x}) && \text{in } \Omega, \\ \varphi(\mathbf{x}) &= \mathscr{U}_\beta\, g(\mathbf{x}) + \varphi_\beta^0(\mathbf{x}) && \text{in } \Omega^+. \end{aligned} \tag{3.12}$$

This satisfies (2.5) and (2.7). By (3.9) and (3.11) it will also satisfy (2.6) provided that f and g satisfy,

$$\begin{aligned} \mu_m G_{\sqrt{i\alpha}} f &= \mu_a G_\beta\, g + \chi_1, \\ -\frac{1}{2}\frac{f}{p} + N_{\sqrt{i\alpha}} f &= \frac{1}{2}\frac{g}{p} + N_\beta\, g + \chi_2, \end{aligned} \tag{I_β}$$

$$\chi_1(\sigma) = \mu_a \varphi_\beta^0(\mathbf{X}(\sigma)), \qquad \chi_2(\sigma) = \frac{\partial}{\partial n}\varphi_\beta^0(\mathbf{X}(\sigma)). \tag{3.13}$$

The procedure for (P_0) needs some modification. Recall that in (P_0) we require that φ be bounded as $\|\mathbf{x}\| \to \infty$. But from (2.10) we see that $\varphi_0^0(\mathbf{x}) \sim (2\pi)^{-1}\log\|\mathbf{x}\|$ as $\|\mathbf{x}\| \to \infty$. Thus the scattered field, $\varphi - \varphi_0^0$ must contain a compensating term. For (P_0) we take,

$$\begin{aligned} \varphi(\mathbf{x}) &= \mathscr{U}_{\sqrt{i\alpha}} f(\mathbf{x}) && \text{in } \Omega, \\ \varphi(x) &= \mathscr{U}_0\, g(\mathbf{x}) + c + \varphi_0^0(\mathbf{x}) && \text{in } \Omega^+, \end{aligned} \tag{3.14}$$

where now f, g and the constant c are to be determined. Condition (2.6) again yields two equations. A third is obtained by noting that $\mathscr{U}_0\, g(\mathbf{x}) \sim Mg \log\|\mathbf{x}\|$ as $\|\mathbf{x}\| \to \infty$. Thus our system is,

$$\begin{gathered} \mu_m G_{\sqrt{i}\alpha} f = \mu_a G_0 g + c + \chi_1, \\ -\frac{1}{2}\frac{f}{p} + N_{\sqrt{i}\alpha} f = \frac{1}{2}\frac{g}{p} + N_0 g + \chi_2, \\ Mg = -1, \end{gathered} \tag{I_0}$$

where χ_1 and χ_2 are as in (3.13) but with φ_0^0. The third condition of (I_0) guarantees that φ remains bounded at infinity.

Remark 3.1. The effectiveness of our method depends crucially on the use of the simple layers (3.6) to represent the solution. This makes possible the simply computed expression (3.11) for the normal derivative.

Remark 3.2. We have, by $(2.5)_2$, $(2.6)_2$ and Green's theorem,

$$\begin{aligned} \int_\Omega\int \varphi(\mathbf{x})\,d\mathbf{x} &= -\frac{1}{i\alpha^2}\int_\Omega\int \Delta\varphi\,d\mathbf{x} = -\frac{1}{i\alpha^2}\int_\Gamma \left(\frac{\partial\varphi}{\partial n}\right)^-\,ds \qquad (3.15) \\ &= -\frac{1}{i\alpha^2}\int_\Gamma \left(\frac{\partial\varphi}{\partial n}\right)^+\,ds = -\frac{1}{i\alpha^2}\left(\int_{\|\mathbf{x}\|=R}\left(\frac{\partial\varphi}{\partial n}\right)ds + \int_{\|\mathbf{x}-\mathbf{x}_0\|=\epsilon}\frac{\partial\varphi}{\partial n}\,ds\right). \end{aligned}$$

Since φ is bounded at infinity $\partial\varphi/\partial n = O(R^{-2})$ on $\|\mathbf{x}\| = R$; hence the limit as $R \to \infty$ of the first term on the right is zero. The second term, however, has the value $1/i\alpha^2$ since $\varphi \sim \varphi_0$ near $\mathbf{x}_0$. Hence (2.14) yields,

$$\mathscr{T}(t) = \operatorname{Re}(-iI_0 e^{-i\omega t}).$$

Thus $(I_0)_3$ is equivalent to the statement that the total current in the cylinder is equal to I_0 in magnitude.

4. The Asymptotic Method

Here we describe an approximate procedure for large α. Our first step is to introduce a local coordinate system (Fig. 2). At each point $\mathbf{x} = \mathbf{X}(\tau)$ of Γ we construct the unit inner normal $\mathbf{n}(\tau) = \{-\dot{X}(\tau)\hat{\imath} + \dot{Y}(\tau)\hat{\jmath}\}/p(\tau)$. Points along this normal are then given by,

$$\mathbf{x} = \chi(\tau, \eta) \equiv \mathbf{X}(\tau) + \eta\mathbf{n}(\tau). \tag{4.1}$$

Equation (4.1) defines a coordinate system in a sufficiently narrow boundary strip $\tilde{\Omega}$. This means that (4.1) can be solved for,

$$\tau = T(\mathbf{x}), \qquad \eta = N(\mathbf{x}) \tag{4.2}$$

in $\tilde{\Omega}$ (η small).

The local coordinate system is orthogonal with the form,

$$Q^2 d\tau^2 + d\eta^2, \qquad Q(\tau, \eta) = (1 - \eta k(\tau)), \tag{4.3}$$

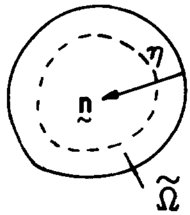

FIGURE 2

where k is the curvature of Γ. If $\mathbf{t}(\tau)$ is the unit tangent, $\mathbf{t}(\tau)=(\dot{X}(\tau)\hat{\imath}+\dot{Y}(\tau)\hat{\jmath})/p(\tau)$, the gradient and Laplacian can be expressed as,

$$\nabla=\mathbf{t}Q^{-1}\frac{\partial}{\partial\tau}+\mathbf{n}\frac{\partial}{\partial\eta};\qquad \Delta=Q^{-1}\left\{\frac{\partial}{\partial\tau}\left(Q^{-1}\frac{\partial}{\partial\tau}\right)+\frac{\partial}{\partial\eta}\left(Q\frac{\partial}{\partial\eta}\right)\right\}. \tag{4.4}$$

We seek an expansion of the solution in the form,

$$\begin{aligned}\varphi(\mathbf{x}) &\sim e^{-s\eta}\sum_{k=1}^{\infty} a_k s^{-k} \qquad \text{in } \tilde{\Omega}\\ &\sim b_0+\sum_{k=1}^{\infty} b_k s^{-k} \qquad \text{in } \Omega.\end{aligned} \tag{4.5}$$

Here $s=\sqrt{-i}\,\alpha$ and the coefficients a_m, b_m are independent of s. Notice that the expansion in (4.5) decreases exponentially as we move into the conductor. This represents the skin effect ($\alpha=\sqrt{2}\,L/D$, where D is the skin depth). In (4.5) one should think of a_m as functions of τ and η and then of $\mathbf{x}$ through (4.2) while the b_m are functions of $\mathbf{x}$.

The coefficients a_k and b_k can be determined recursively by substituting (4.5) into (2.5) and (2.6) and equating coefficients of like powers of s. $(2.5)_1$, $(2.6)_1$ and (4.5) yield immediately,

$$\Delta b_k=-\beta^2 b_k,\qquad (\Delta b_m=0 \text{ if } \beta=0), \tag{4.6}$$

$$b_0(\mathbf{X}(\tau))\equiv 0,\qquad \mu_a b_k(\mathbf{X}(\tau))=\mu_m a_k(\tau,0),\qquad k\geqslant 1. \tag{4.7}$$

Noting that differentiation in the outer normal direction is the same as $-\partial/\partial\eta$ we find from $(2.6)_2$,

$$\begin{aligned}a_1(\tau,0)&=\frac{\partial b_0}{\partial n}(\mathbf{X}(\tau))^+;\\ a_{k+1}(\tau,0)&=\frac{\partial b_k}{\partial n}(\mathbf{X}(\tau))^+ +\frac{\partial a_k}{\partial\eta}(\tau,0),\qquad k\geqslant 1.\end{aligned} \tag{4.8}$$

The calculation for $(2.5)_2$ is more complicated. It is facilitated by noting the formulas

$$\begin{gathered}\operatorname{grad} e^{-s\eta} = -se^{-s\eta}\mathbf{n}, \qquad \Delta e^{-s\eta} = s^2 e^{-s\eta} - \frac{sQ_\eta}{Q} e^{-s\eta}, \\ \Delta(fg) = \Delta f + 2\nabla f \cdot \nabla g + f\Delta g.\end{gathered} \tag{4.9}$$

With these formulas $(2.5)_1$ will be found to yield

$$\begin{gathered}2\frac{\partial a_1}{\partial \eta} + \frac{Q_\eta}{Q} a_1 = 0, \\ 2\frac{\partial a_k}{\partial \eta} + \frac{Q_\eta}{Q} a_k = \Delta a_{k-1}, \qquad k \geqslant 2.\end{gathered} \tag{4.10}$$

Equations (4.10) are ordinary differential equations with τ as a parameter. They may be integrated in the form ($Q(\tau, 0) \equiv 1$),

$$\begin{gathered}a_1(\tau, \eta) = Q(\tau, \eta)^{-1/2} a_1(\tau, 0), \\ a_k(\tau, \eta) = Q(\tau, \eta)^{-1/2} \left\{ a_1(\tau, 0) + \tfrac{1}{2} \int_0^\eta Q(\tau, \xi)^{-1/2} \Delta a_{k-1}(\tau, \xi)\, d\xi \right\}.\end{gathered} \tag{4.11}$$

Remark 4.1. It will be recognized that our procedure is completely analogous to geometrical optics and formulas (4.10)–(4.11) are familiar in that context.

Our formulas can be used recursively. Observe that (4.6) and (4.7) show that b_0 satisfies,

$$\Delta b_0 = -\beta^2 b_0(0) \quad \text{in } \Omega^+, \qquad b_0 = 0 \quad \text{on } \Gamma \text{ for } \beta > 0\ (=0). \tag{4.12}$$

It must be noted that b_0 must also carry the singular term $\varphi_\beta^0(\varphi_0^0)$. Thus b_0 is the solution of an exterior Dirichlet problem and is the infinite conductivity approximation.

Let us consider the exterior Dirichlet problem,

$$\begin{gathered}\Delta v = -\beta^2 v(0) \quad \text{in } \Omega^+, \qquad v = \nu \quad \text{on } \Gamma. \\ v \text{ satisfies a radiation condition } (v \text{ bounded as } \|\mathbf{x}\| \to \infty),\end{gathered} \tag{4.13}$$

for $\beta > 0$ ($\beta = 0$). Denote the solution by $V(\mathbf{x} : \nu)$.

Now we proceed as follows. Solve (4.12) for b_0 and compute $(\partial b_0/\partial n)^+$. Then (4.8) yields $a_1(\tau, 0) = (\partial b_0/\partial n)^+$ and (4.11) determines $a_1(\tau, \eta)$. Also (4.7) yields,

$$b_1(\mathbf{X}(\tau)) = \xi a_1(\tau, 0) = \xi \left(\frac{\partial b_0}{\partial n}\right)^+ \equiv \nu_1(\tau), \qquad \xi = \mu_m/\mu_a.$$

It follows from (4.6) that $b_1(\mathbf{x}) = V(\mathbf{x}, \nu_1)$. Compute $(\partial b_1/\partial n)^+$ and then (4.8) yields

$a_2(\tau, 0)$ and (4.11) $a_2(\tau, \eta)$. Moreover from (4.7), $b_2(\mathbf{X}(\tau)) = \xi a_2(\tau, 0) \equiv \nu_2(\tau)$ and hence $b_2(\mathbf{x}) = V(\mathbf{x}, \nu_2)$.

Clearly the process can be continued to get the b_k and a_k recursively. It is useful to observe that if one only wants the first three terms of the exterior field then the intermediate steps of computing the a_k's can be eliminated. Indeed if we let $\eta \to 0$ in (4.10), we find,

$$\frac{\partial a_1}{\partial n}(\tau, 0) = -Q_\eta(\tau, 0)\, a_1(\tau, 0)/2 = -Q_\eta(\tau, 0) \left(\frac{\partial b_0}{\partial n}\right)^+ \Big/ 2.$$

Hence by (4.8) and (4.6),

$$\nu_2(\tau) = b_2(\mathbf{X}(\tau)) = \xi a_2(\tau, 0) = \left(\frac{\partial b_1}{\partial n}\right)^+ - Q_\eta(\tau, 0) \left(\frac{\partial b_0}{\partial n}\right)^+ \Big/ 2.$$

We can solve problem (4.12) by a simplified version of the procedure in Section 3. Consider first $\beta > 0$. For (4.12) we put

$$b_0(\mathbf{x}) = \mathscr{U}_\beta\, g(\mathbf{x}) + \varphi_\beta^0(x), \qquad G_\beta\, g + \varphi_\beta^0 = 0 \qquad \text{on } \Gamma. \tag{4.14}$$

For (4.13),

$$v(\mathbf{x}) = \mathscr{U}_\beta\, g(\mathbf{x}), \qquad G_\beta\, g = \nu \qquad \text{on } \Gamma. \tag{4.15}$$

For $\beta = 0$ and (4.12) we have,

$$\begin{gathered} b_0(\mathbf{x}) = \mathscr{U}_0\, g(x) + C + \varphi_0^0(\mathbf{x}); \\ G_0\, g + C + \varphi_0^0 = 0 \qquad \text{on } \Gamma, \qquad Mg = -1 \end{gathered} \tag{4.16}$$

and for $\beta = 0$ and (4.13),

$$v(\mathbf{x}) = \mathscr{U}_0\, g(x) + C; \qquad G_0\, g + C = \nu, \qquad Mg = 0. \tag{4.17}$$

It will follow from the results of Section 6 that the integral equations in (4.14)–(4.17) all have solutions. Note also that with our choice of representation of the solutions the normal derivatives are easily computed by means of (3.11).

Remark 4.2. We conjecture but have not completely proved that (4.5) gives valid asymptotic expansions in $\tilde{\Omega}$ and Ω^+. Inside the inner boundary of $\tilde{\Omega}$, φ should be exponentially small.

5. Numerical Implementation

We indicate a procedure for the approximate numerical solution of integral equations (I_β) and (I_0) as well as those in (4.14)–(4.17). We introduce a uniform grid

$$\tau_j = jh, \qquad j = 0, 1, \ldots, N-1, \qquad h = 2\pi/N$$

on the parameter interval. We introduce the mapping $H: \mathscr{F} \to R^N$ which takes f into its values at the grid points

$$(Hf)_j = f(\tau_j). \tag{5.1}$$

For (I_β) our goal is to obtain approximations to Hf and Hg, f and g the solution. We will approximate the integrals by rectangular quadrature. For any continuous kernel $k(\sigma, \tau)$, rectangular quadrature gives

$$HKf \approx \mathscr{K}Hf, \qquad \mathscr{K}_{ij} = hk(\tau_i, \tau_j), \qquad i, j = 0, \dots, N-1. \tag{5.2}$$

For kernels with logarithmic singularities we have to modify this formula. It is not difficult to verify the formula,

$$\int_0^{2\pi} \ell(\sigma, \tau)\, d\tau = \frac{1}{2\pi} \int_0^{2\pi} \log \sin \left| \frac{\sigma - \tau}{2} \right| d\tau \equiv \delta = -\log 2. \tag{5.3}$$

This suggests the approximate formula,

$$\begin{aligned}(HLf)_i &= \delta f(\tau_i) + \int_0^{2\pi} (f(\tau) - f(\tau_i))\, \ell(\tau_i, \tau)\, d\tau \\ &\approx \delta (Hf)_i + h \sum_{\substack{j=0 \\ j \neq i}}^{N-1} ((Hf)_j - (Hf)_i)\, \ell(\tau_i, \tau_j).\end{aligned} \tag{5.4}$$

Our procedure, then, is simply to replace all the integrals occurring in (I_β) by their approximate values using either (5.2) or (5.4) and replace Hf and Hg by approximate values $\mathbf{F}$ and $\mathbf{G}$. In (I_0) we approximate the last condition as $h \sum_{j=0}^{N-1} G_j = -1$. This yields $2N$ equations for $\mathbf{F}$ and $\mathbf{G}$ or $2N+1$ for $\mathbf{F}$ and $\mathbf{G}$ and C and these equations are then solved.

The same procedure can be applied to (4.14)–(4.17) and we note two major simplifications. First at each step we solve only N or $N+1$ equations. Secondly, the process of assembling the matrices for the left-hand sides of (I_β) or (I_0) is eliminated. This is a fairly complicated process since it involves the evaluation of the Bessel functions $H_0^{(1)}(\sqrt{i}\, \alpha\rho)$. This evaluation can, if necessary, be carried out by using approximation formulas given in [1].

Remark 5.1. As a part of our process we compute φ on Γ. The calculation of φ close to but not on Γ from (3.12) or the analogous formulas is, however, likely to be unstable due to the presence of the logarithmic term in G_j. We point out, however, that φ can be obtained for boundary strips $\tilde{\Omega}_-$ and $\tilde{\Omega}_+$ as in Fig. 3. The values on Γ can be computed and also the values of φ on the inner boundary of $\tilde{\Omega}_-$ and the outer boundary of $\tilde{\Omega}_+$ can be found from (3.12). Thus φ satisfies Dirichlet problems in $\tilde{\Omega}_-$ and $\tilde{\Omega}_+$ which can be solved by standard methods, finite differences or finite elements. In $\Omega/\tilde{\Omega}_-$ and in $\Omega^+/\tilde{\Omega}_+$ one can use (3.12). A similar idea is discussed in [5].

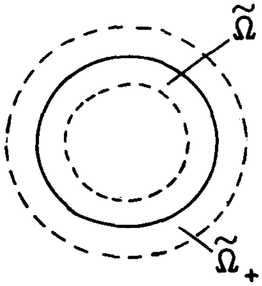

FIGURE 3

In order to test the validity of our procedures we performed some sample calculations. We considered the case of a circular cylinder. In this case infinite series solutions of (P_β) and (P_0) can be obtained by separation of variables. We computed approximate values for the first two terms in this series by both our methods. We present the results in the tables.

Table I gives an indication of the accuracy with decreasing grid size. We indicate the exact value of $|\varphi|$ on Γ (Ex $|\varphi|$) at the points $\bar{\tau}_k = \pi k/5$, $k = 0, 1,..., 5$. The solution is of course symmetric about $y = 0$. Table I is the case $\beta = 0$ with $\alpha = \sqrt{10}$ and the ratio of the distance of the wire from the center to the radius was 2.5. We give the absolute value of the errors for 20, 30 and 40 mesh points. The L_1 errors are the average errors over the interval. The errors in real and imaginary parts were approximately the same.

Remark 5.2. It will be seen that the errors are approximately of order h^3. We have not completed the error analysis but preliminary results indicate that this is correct theoretically. This is rather surprising in view of the crude quadrature rules we employed. It rests on the use of the uniform grid and the fact that rectangular quadrature for periodic functions with a uniform grid is very accurate. A similar phenomenon was observed in [6]. We observe that this is one order of accuracy higher than comes from finite element methods which means that one can obtain the same accuracy with our method with fewer equations.

TABLE I

k	$\lvert \text{Ex}\,\varphi \rvert$	Err $N = 20$	Err $N = 30$	Err $N = 40$
0	0.2868	0.0093	0.0027	0.0011
1	0.2634	0.0087	0.0026	0.0010
2	0.2033	0.0069	0.0020	0.0009
3	0.1305	0.0046	0.0014	0.0006
4	0.0720	0.0030	0.0008	0.0004
5	0.0506	0.0024	0.0006	0.0003
L_1 errors		0.0058	0.0017	0.0007

TABLE II

	$\alpha=\sqrt{1}$		$\alpha=\sqrt{3}$		$\alpha=\sqrt{5}$		$\alpha=\sqrt{10}$	
k	$\lvert\mathrm{Ex}\,\varphi\rvert$	$\lvert\mathrm{Err}\rvert$	$\lvert\mathrm{Ex}\,\varphi\rvert$	$\lvert\mathrm{Err}\rvert$	$\lvert\mathrm{Ex}\,\varphi\rvert$	$\lvert\mathrm{Err}\rvert$	$\lvert\mathrm{Ex}\,\varphi\rvert$	$\lvert\mathrm{Err}\rvert$
0	0.8721	0.0880	0.5222	0.0071	0.4055	0.0033	0.2868	0.0012
1	0.8150	0.0948	0.4821	0.0078	0.3741	0.0033	0.2634	0.0013
2	0.6717	0.1150	0.3773	0.0100	0.2930	0.0037	0.2038	0.0015
3	0.5170	0.1425	0.2878	0.0127	0.1912	0.0041	0.1305	0.0019
4	0.4276	0.1501	0.1515	0.0149	0.1121	0.0044	0.0720	0.0021
5	0.4070	0.1054	0.1179	0.0158	0.0837	0.0046	0.0506	0.0021
L_1 errors		0.1160		0.0113		0.0039		0.016

Table II gives an indication of the accuracy of the asymptotic approximation. We tabulate the same quantities, again for $\beta=0$, but for four different values of α. Our approximate values of $|\varphi|$ on Γ were computed by using the first three terms of the expansion (4.5) in Ω^+ and then numerically implementing the resulting problems.

We observe that the asymptotic procedure is not too bad even for the value $\alpha=1$ and its accuracy increases markedly with increasing $\alpha=1$.

6. Proofs

In this section we give a very brief outline of the proofs of validity of our procedures. The technical details are quite complicated and appear in [4]. Here we present only the key ideas. For further simplicity we assume $\mu_m=\mu_a$.

THEOREM 1. *Problems (P_β), $\beta\geqslant 0$, have at most one solution.*

Proof. We introduce some notation. For any region ω and curve γ and any function z we write,

$$|z|_0^2(\omega)=\int_\omega |z|^2\,dx,\qquad |z|_1^2(\omega)=\int_\omega |\nabla z|^2\,dx,$$
$$\Lambda(z)(\gamma)=\int_\gamma \bar z\,\frac{\partial z}{\partial n}\,ds. \tag{6.1}$$

We choose R so large that $B_R(0)=\{\mathbf{x}\colon \|\mathbf{x}\|<R\}\supset\Omega$ and put $\gamma_R=\partial B_R(0)$, $\Omega_R^+=B_R(0)\cap\Omega^+$.

In (P_β), $\beta\geqslant 0$, we need only show that the solution φ for $\varphi_\beta^0\equiv 0$ is zero. Let φ be such a solution and consider first (P_0). Green's theorem applied first to Ω and then to Ω_R^+ yields,

$$\begin{gathered}\operatorname{Im} \Lambda(\varphi^-)(\Gamma) = -\alpha^2\,|\varphi|_0^2(\Omega), \qquad \operatorname{Re} \Lambda(\varphi^-)(\Gamma) = |\,\varphi\,|_1^2(\Omega),\\ \Lambda(\varphi)(\gamma_R) - \Lambda(\varphi^+)(\Gamma) - |\varphi|_1^2(\Omega_R^+) = 0.\end{gathered} \tag{6.2}$$

The interface conditions yield $\Lambda(\varphi^-)(\Gamma) = \Lambda(\varphi^+)(\Gamma)$ and we conclude,

$$\operatorname{Im} \Lambda(\varphi^-)(\Gamma) = \operatorname{Im} \Lambda(\varphi^+)(\Gamma) = \operatorname{Im} \Lambda(\varphi)(\gamma_R). \tag{6.3}$$

But φ is bounded at infinity. Hence $\nabla\varphi = O(R^{-2})$ on γ_R; hence we can let $R \to \infty$ in (6.3) and conclude that $\operatorname{Im} \Lambda(\varphi^-)(\Gamma) = 0$. Then the first of Eqs. (6.2) yields $\varphi \equiv 0$ in Ω. It follows that $\Lambda(\varphi^-)(\Gamma) = \Lambda(\varphi^+)(\Gamma) = 0$ and passing to limit $R = \infty$ in the last of (6.2) yields $\varphi \equiv 0$ in Ω^+.

When $\beta > 0$ the last of Eqs. (6.2) is replaced by

$$\Lambda(\varphi)(\gamma_R) - \Lambda(\varphi^+)(\Gamma) - |\varphi|_1^2(\Omega_R^+) + \beta^2\,|\varphi|_0^2(\Omega_R^+) = 0. \tag{6.4}$$

The imaginary part of (6.4) is,

$$\operatorname{Im} \Lambda(\varphi^-)(\Gamma) = \operatorname{Im} \Lambda(\varphi^+)(\Gamma) = \operatorname{Im} \Lambda(\varphi)(\gamma_R).$$

A standard argument shows that the radiation condition implies that the limit of the term on the right as $R \to \infty$ is zero. Hence we conclude as before that $\operatorname{Im} \Lambda(\varphi^-)(\Gamma) = 0$ and hence $\varphi \equiv 0$ in Ω. Now the usual uniqueness theorem for the exterior Dirichlet problem for $\Delta\varphi + \beta^2\varphi = 0$ implies $\varphi \equiv 0$ in Ω^+.

COROLLARY. *Equation (I_0) has at most one solution. If Ω is such that $\Delta\varphi = -\beta^2\varphi$ in Ω, $\varphi \equiv 0$ on Γ implies $\varphi \equiv 0$. Then (I_β), $\beta > 0$ has at most one solution.*

Proof. Again this is a standard argument. The existence of a solution of either set with $\chi_1 = \chi_2 \equiv 0$ is easily seen to produce, via (3.12) or (3.14), a solution of the homogeneous boundary value problem (P_β) or (P_0) which must be zero. This means that $\mathscr{U}_{\sqrt{i}\alpha}\, g \equiv 0$ in Ω and $\mathscr{U}_\beta\, g$ or $\mathscr{U}_0\, g + C$ is identically zero in Ω^+. Continuation of these functions to the complementary domains and use of the jump relations and uniqueness theorems for the Dirichlet problems then implies $f \equiv g \equiv 0$, $C = 0$.[3]

THEOREM 2. *The system (I_0) has a solution.*

THEOREM 3. *Suppose[4] that the region Ω is such $\Delta\varphi = -\beta^2\varphi$ in Ω, $\varphi \equiv 0$ on Γ implies that $\varphi \equiv 0$ in Ω. Then (I_β) has a solution.*

[3] See the proof of Theorem 3 below.

[4] This condition on Ω is familiar in diffraction theory.

The idea of the proof of these theorems is conceptually simple. For (I_0) we invert the first and third equations to obtain g in terms of f in the form,

$$g = f + K[f] + \phi_1, \tag{6.5}$$

where ϕ_1 is known and K is an integral operator. Substitution of (5.5) into the second equation yields a Fredholm equation of second kind for f. The corollary implies that the corresponding homogeneous equation has only the zero solution. For (I_β) one obtains (6.5) by inverting the first equation.

The above program is technically very complicated and rests on the theory of singular integral equations as described in [11, 14]. Again details are in [4] and we present only the basic ideas.

We need a considerable amount of additional notation. First we observe that integral operators can be composed. Thus if $K[f] = k \wedge f$ and $J[f] = j \wedge f$ we will have $KJ[f] = (k \wedge j) \wedge f$, where

$$(k \wedge j)(\sigma, \tau) = \int_0^{2\pi} k(\sigma, \xi)\, j(\xi, \tau)\, d\xi. \tag{6.6}$$

We recall the operator L, $Lf = \ell \wedge f$ of Section 2. We need here its derivative. We set

$$D = \frac{d}{d\sigma}, \qquad t(\sigma, \tau) = \frac{1}{4\pi} \cot \frac{\sigma - \tau}{2}, \qquad Tf = t \wedge f \tag{6.7}$$

so that $T = DL$. Finally we need the operator π which subtracts from f its mean value

$$\pi f(\sigma) = f(\sigma) - Mf. \tag{6.8}$$

The basic result in singular integral equation theory is called *Hilbert's formula* ([10, p. 122]). It is,

$$-4T^2 f = \pi f. \tag{6.9}$$

We recall that $G_\gamma = L + R_\gamma$, where R_γ has a smooth kernel. If we combine this result with (6.7) and (6.9) we obtain the main formula we use

$$\begin{aligned} -4TDG_\gamma f &= -4TDLf - 4TDR_\gamma f = -4T^2 f - 4TDR_\gamma f, \\ &= \pi f - 4TDR_\gamma f = f - (M + 4TDR_\gamma) f \equiv f + J_\gamma f. \end{aligned} \tag{6.10}$$

One can show (see [14]) that the composition of T with the kernel R_γ yields an integral operator with a kernel which is continuous and hence J_γ is continuous.

Our goal is to use (6.10) to obtain formula (6.5). To do this we must discuss the problem of inverting the first and third equations of (I_0) or the first of (I_β). Both cases are covered by the following fundamental result, the proof of which is presented at the end of the section.

LEMMA. (i) *There exists a unique function* g_0 *and constant* c_0 *such that,*

$$G_0 g_0 + c_0 = 0, \qquad Mg_0 = 1. \tag{6.11}$$

(ii) *There exists a kernel* $p(\sigma, \tau)$ *such that,*

$$p(\sigma, \tau) = -4t(\sigma, \tau) + q(\sigma, \tau), \qquad q \text{ smooth}, \tag{6.12}$$

$$\pi G_0 PD = \pi. \tag{6.13}$$

Consider the integral equation problem,

$$G_0 g + c = h, \qquad Mg = -1. \tag{6.14}$$

We seek a solution of (6.14) in the form $g = PDh + \eta g_0$. From $(6.11)_2$ we see that $(6.14)_2$ will be satisfied if $\eta = -1 - MPDh$. With η so chosen, (6.13) and $(6.11)_1$ yield

$$\begin{aligned} G_0 g &= G_0 PDh + \eta G_0 g_0 = \pi G_0 PDh + MG_0 PDh - \eta c_0 \\ &= \pi h + MG_0 PDh - \eta c_0 = h + \{MG_0 PDh - \eta c_0 - Mh\}. \end{aligned}$$

Thus we satisfy $(6.14)_1$ by choosing $c = MG_0 PDh - \eta c_0 - Mh$. Collecting the results we have a solution of (6.14) in the form,

$$g = PDh - (MPDh) g_0 - g_0, \qquad -c = MG_0 PDh - Mh + c_0(1 + MPDh). \tag{6.15}$$

If we combine $(6.15)_1$ with (6.12) we see that our solution can be written,

$$g = -4TDh + QDh - (MPDh) g_0 - g_0. \tag{6.16}$$

We can now indicate the proof of Theorem 2. Suppose we have a solution. If we set $h = G_{\sqrt{i}\alpha}[f] - \chi_1$ we have (6.14). Thus we can apply (6.16) and (6.10) to obtain an equation of the form (6.5), where

$$\begin{aligned} Kf &= J_\gamma f + QDG_{\sqrt{i}\alpha} f - (MPDG_{\sqrt{i}\alpha} f) g_0, \\ \phi_1 &= 4TD\chi_1 - QD\chi_1 + (MPD\chi_1) g_0 - g_0. \end{aligned} \tag{6.17}$$

Substitution of (6.16) into $(I_0)_2$ yields the desired second kind equation for f. It is shown in [4] that the kernel of the second kind equation is in L_2 so that Fredholm theory applies and one obtains a solution f. Reversing the steps yields g and c and one has a solution of (I_0).

The treatment of (I_β) is more complicated still. Our goal is to invert $(I_\beta)_1$ which means solving the first kind equation,

$$G_\beta g = h, \qquad \beta > 0. \tag{6.18}$$

We perform this task in two steps. First we observe that because of (2.8) we can write,

$$G_\beta g = G_0 g + \lambda_0 Mg + K_\beta g, \qquad K_\beta g = k_\beta \wedge g. \tag{6.19}$$

The kernel $k_\beta(\sigma, \tau)$ is continuously differentiable and $\operatorname{Im} \lambda_0 \neq 0$.

Consider now the problem

$$G_0 g + \lambda_0 Mg = h. \tag{6.20}$$

We can solve this, just as we did (6.14), in the form $g = PDh + \xi g_0$. We have then by (6.11) and (6.13),

$$G_0 g + \lambda_0 Mg = h + \{MG_0PDh - Mh + \lambda_0 MPDh\} + (\lambda_0 - c_0)\xi.$$

Since $\operatorname{Im} \lambda_0 \neq 0$ the constant $\lambda_0 - c_0$ is non-zero and we can choose ξ so as to make the last two terms zero. Thus we have a solution of (6.20) in the form $g = PDh + \xi(PDh)\, g_0$, where ξ is a linear functional of h.

We use our last result to invert G_β. Suppose $G_\beta g = h$ then by the results of the last paragraph and (6.19) we have,

$$g = -PDK_\beta g - \xi(PDK_\beta g)\, g_0 + PDh + \xi(PDh)\, g_0. \tag{6.21}$$

Equation (6.21) is again a second kind equation with bounded kernel. If the corresponding homogeneous equation has a solution $\tilde{g}$ then one can verify that $\mathscr{U}_\beta \tilde{g}$ would be a solution of $\Delta u + \beta^2 u = 0$ in Ω^+ vanishing on Γ and satisfying a radiation condition. Hence $u \equiv 0$ in Ω^+. Consider then $\mathscr{U}_\beta \tilde{g}$ in Ω. It is again a solution of $\Delta \varphi + \beta^2 \varphi = 0$ with $\varphi \equiv 0$ on Γ. By our hypothesis on Ω it follows that $\mathscr{U}_\beta \tilde{g} \equiv 0$ in Ω. Then $\tilde{g} \equiv 0$ since it is the difference of the normal derivatives from inside and outside. Hence (6.21) has a solution.

We have shown that G_β has an inverse, the leading term of which is the operator PD just as for G_0 together with $(I_0)_3$. Thus we can proceed just as in the proof for (I_0), expressing g in terms of f and substituting into $(I_\beta)_2$ to get an equation for f which is once more solvable by our Corollary to Theorem 1.

We have still to indicate the proof of the Lemma. Suppose g_0 is a solution of (6.11) and set $U_0 = \mathscr{U}_0 g_0$. Then $\Delta U_0 = 0$ in Ω with $U_0 = -C_0$ on Γ so $U_0 \equiv -C_0$ in Ω. Also we have $\Delta U_0 = 0$ in Ω^+, $U_0 = -C_0$ on Γ and,

$$U_0 = Mg_0 \log r + O(r^{-1}) = \log r + O(r^{-1}) \qquad \text{as } r = |x| \to \infty. \tag{6.22}$$

Now let us map Ω^+ onto $|w| > R$ with $w = \zeta(z)$ such that $\zeta(z) = z + O(1)$ as $z \to \infty$. Such a map exists and is unique by the Riemann mapping theorem. Let $z = \psi(w)$ be the inverse and set $\mathscr{U}(w) = U_0(\psi(w))$. Then $\mathscr{U}$ satisfies,

$$\begin{gathered} \Delta\mathscr{U} = 0 \quad \text{in } |w| > R, \qquad \mathscr{U} = -c_0 \quad \text{on } |w| = R, \\ \mathscr{U} = \log|w| + O(|w|^{-1}) \quad \text{as } |w| \to \infty. \end{gathered} \tag{6.23}$$

The only solution of (6.23) is $\mathscr{U} = \log|w|$, $c_0 = -\log R$.

It follows from the above the the function U is given in Ω by

$$U_0 = \log|\zeta(z)|, \qquad c_0 = -\log R. \tag{6.24}$$

Recall that we had $U \equiv -C_0$ in Ω. Suppose $p \equiv 1$. Then from (6.24) and the jump relation.

$$g_0 = \left(\left(\frac{\partial U_0}{\partial n}\right)^+ - \left(\frac{\partial U_0}{\partial n}\right)^-\right) = \frac{\partial}{\partial n}(\log|\zeta(z)|). \tag{6.25}$$

If g_0 is a solution of (6.11), it must have the form (6.25). Arguing in the other direction we can conclude that (6.25) does indeed solve (6.11). The case where p is not one can be handled by a minor variation. This proves (i).

To establish (ii) we need ideas from [14]. Consider the integral equation

$$DG_0\, g = H. \tag{6.26}$$

Since g_0 differs from ℓ be a regular kernel and $DL = T$ we see that (6.26) is a singular integral equation with index zero (see [14]). It is shown in [14] that the Fredholm alternative holds for such equations. Furthermore, when the necessary conditions for existence are met there is a resolvent solution of the form $g = PH$, where P has the form (6.12).

In order to apply the above theory we must investigate the null spaces of DG_0 and its adjoint. The first observation is that the adjoint homogeneous equations has the form,

$$(DG_0)\,\chi(\sigma) = \int_0^{2\pi} \frac{\partial}{\partial \tau}\, g_0(\sigma, \tau)\,\chi(\tau)\, d\tau = 0.$$

One solution is clearly $\chi(\tau) \equiv 1$ so $\dim \eta((DG_0)^*) \geqslant 1$. We show it is exactly one.

Suppose $DG_0\, g \equiv 0$. Then, integrating, we have $G_0\, g = k$, a constant. If $Mg = \delta$ then it follows from (i) of the Lemma that $g = \delta g_0$. Thus $\dim \eta(DG_0) = 1$. It follows that constants are the only solutions of $(DG_0)^* h = 0$.

Now consider the equation $G_0\, g = h$. Differentiating we obtain (6.26) with $H = Dh$. But then $\int_0^{2\pi} H\, ds = 0$ so the necessary condition of the Fredholm alternative apply and we will have a solution $g = PH = PDh$ of (5.26). Integrating, we have,

$$G_0 PDh = h + k, \tag{6.27}$$

where k is a constant. If we average we obtain $M(G_0 PDh - h) = k$ and substitution into (6.27) yields (6.13).

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